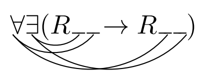
The Syntax and Semantics of Wire Diagrams: Two Approaches

Bryan Pickel[1] and Brian Rabern[2]

[1] University of Glasgow, UK
bryan.pickel@glasgow.ac.uk
[2] University of Edinburgh, UK
brian.rabern@ed.ac.uk

The language of first-order logic standardly makes use of the notation of quantifiers and variables. This notation is used to express statements of multiple generality, such as $\forall x \exists y(Ryx \to Rxy)$. An alternative notation—originally developed independently by Quine and Bourbaki in [12] and [1]—makes use of *wire diagrams*, rendering a quantified formula such as $\forall x \exists y(Ryx \to Rxy)$ as follows.[3]



In a wire diagram, the variables disappear altogether. The link between a quantifier and the argument position into which it binds is instead represented by a "wire" that connects them.

It has recently been argued that this wire-diagram notation is superior to the quantifier-variable notation because it can preserve compositionality without positing overly fine-grained meanings. (See: [5], ([2]: 14), ([13]: 215), and ([11], 38).) Consider the fact that the open sentences (1) and (2) differ semantically.

(1) Rxx

(2) Rxy

Specifically, (1) and (2) embed differently: $\forall x \exists y(Ryx \to Rxx)$ may be false while $\forall x \exists y(Ryx \to Rxy)$ is true.

- COMPOSITIONALITY: If $\alpha = \eta(\beta_1, \ldots, \beta_n)$ and $\gamma = \eta(\delta_1, \ldots, \delta_n)$, then if $[\![\beta_i]\!] = [\![\delta_i]\!]$ (for all i), then $[\![\alpha]\!] = [\![\gamma]\!]$. ([8])
- Formulae (1) and (2) differ semantically: $[\![(1)]\!] \neq [\![(2)]\!]$
- $(1) = \rho(R, x, x)$ and $(2) = \rho(R, x, y)$, where ρ is the formation rule that combines an n-ary predicate with n terms.
- Therefore, 'x' and 'y' differ semantically, or $[\![x]\!] \neq [\![y]\!]$

Among other ills, this approach seems to make the semantic differences between variables objectionably fine-grained and even typographic (see [5]).

Advocates of wire-diagrams suggest that these problems can be avoided by disavowing explicit variables in the notation.

[3] Peirce's diagrammatic representation of quantificational logic in terms of "existential graphs" shares this feature of linking argument positions with wires. See Peirce (1903) *A Syllabus of Certain Topics of Logic*, pp. 15-23.

> ...if we adopt the Quine-Bourbaki notation, then we will not even be able to *ask* whether typographically distinct variables like 'x' and 'y' have different 'semantic roles' ([2]: 14)

However, the syntactic formation and semantic evaluation rules for wire diagrams are rarely developed explicitly. Drawing on research from [9] and [10], this talk develops and compares two approaches. One approach introduces the wires into a formula by a transformation rule (in the sense of [3]). The other approach derives first-order formulae in the standard way, and wires are treated as an additional syntactic feature or as a contextual parameter against which a formula is evaluated.

The transformational approach is strongly compositional, so that the meaning of a complex is a function of the meanings of its immediate constituents, the expressions from which it syntactically derives. It is difficult to determine whether the additional input approach is strongly compositional, because the approach makes the notion of an immediate syntactic constituent unclear. However, it is *weakly compositional*.

1 Transformational Approaches

The transformational approach essentially mimics the syntactic derivation and semantic evaluation in the language of first-order logic. The language of standard first-order logic has a stock of n-ary predicates (F_1^n,F_2^n,...) and terms, including constants (a, b,...) and variables (x_1, x_2,...). An operation ρ combines an n-ary predicate and n terms to form an atomic sentence.

An operation $\wedge$ takes two sentences ϕ and ψ to their conjunction, $\wedge(\phi, \psi)$. An operation, $\neg$ takes a sentence ϕ to its negation $\neg\phi$. Quantified sentences are derived from other sentences in two steps. An operation λ forms a predicate out of a sentence variable ν and a sentence ϕ, $\lambda(\nu, \phi)$. A final rule $\forall$ attaches a quantifier to a derived predicate. Thus a quantified formula such as $\forall xRxb$ can be derived as $\forall\lambda(x, \rho(R, x, b))$.

If the language is compositional, each syntactic operation η (including ρ, $\wedge$, $\neg$, λ, and $\forall$) will be associated with a function f_η that maps the semantic values of its input expressions to the semantic value of its output expression. Thus, $[\![\forall xRxb]\!] = f_\forall f_\lambda([\![x]\!], f_\rho([\![R]\!], [\![x]\!], [\![b]\!]))$.

On the alternative approach, the variable binding operation $\lambda(\nu, \phi)$ removes occurrences of the variable ν from ϕ and connects them with wires which are then connected to the initial $\forall$.

$$\forall\lambda(x, \rho(R, x, b)) = \forall R\_\, b$$

While this approach may be visually illuminating, it does not help with the problem of fine-graining the variables. For, if the operation is to be compositional, then the syntactic operations must still each correspond to a semantic operation. Thus, $[\![\forall\lambda(x, \rho(R, x, b))]\!] = f_\forall f_\lambda([\![x]\!], f_\rho([\![R]\!], [\![x]\!], [\![b]\!]))$. Consider again (1) Rxx

and (2) Rxy. These must differ semantically because they figure in the syntactic derivation of $\forall x \exists y(Ryx \to Rxx)$ and $\forall x \exists y(Ryx \to Rxy)$, respectively. But since $(1) = \rho(R, x, x)$ and $(2) = \rho(R, x, y)$, it follows by compositionality that $[\![x]\!] \neq [\![y]\!]$.

2 Additional Input Approaches

The alternative approach treats the wires as additional inputs to semantic processing. For example, Fine ([4]: 628) conceives of the coordination scheme as syntactic in nature: "the syntactic object of evaluation will no longer be a sequence of expressions but a coordinated sequence of expressions". Alternatively, the wire might be taken as an auxiliary input to semantic processing. In either implementation, the wires are treated as an equivalence relation on the occurrences of variables ([5]: 30). See [7] and [6] for related discussion of Bourbaki's syntax.

Pickel and Rabern formalize Fine's approach to wire diagrams ([9]). It is shown that the approach is *weakly compositional* so that if two formulae differ semantically, then their terminal constituents or mode of combination must differ semantically. However, the notion of immediate constituent—therefore strong compositionality—becomes difficult to assess.

We conclude that wire diagrams do not have the alleged semantic advantages over the purely symbolic notation of quantifiers and variables.

References

1. Bourbaki, N.: Éléments de mathématique: Théorie des ensembles. Paris, Hermann (1954)
2. Button, T., Walsh, S.: Philosophy and Model Theory. Oxford University Press, Oxford (2018)
3. Chomsky, N.: Syntactic Structures. The Hague/Paris: Mouton (1957)
4. Fine, K.: The role of variables. The Journal of Philosophy **100**(12), 605–631 (2003)
5. Fine, K.: Semantic relationism. Blackwell Publishing (2007)
6. Grimm, J.: Implementation of Bourbaki's Elements of Mathematics in Coq: Part One, Theory of Sets. Research Report RR-6999, INRIA (2013)
7. Mathias, A.R.D.: A term of length 4 523 659 424 929. Synthese **133**(1/2), 75–86 (2002)
8. Pagin, P., Westerståhl, D.: Compositionality I: Definitions and Variants. Philosophy Compass **5**(3), 250–264 (2010)
9. Pickel, B., Rabern, B.: Does semantic relationism solve Frege's puzzle? Journal of Philosophical Logic **46**(1), 97–118 (2017)
10. Pickel, B., Rabern, B.: Against Fregean Quantification (unpublished manuscript)
11. Potter, M.: The Rise of Analytic Philosophy, 1879–1930: From Frege to Ramsey. Routledge (2020)
12. Quine, W.V.: Mathematical Logic, Revised Edition. Harvard University Press (1940/1981)
13. Wehmeier, K.F.: The proper treatment of variables in predicate logic. Linguistics and Philosophy **41**(2), 209–249 (2018)